Indeterminate Forms

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, with $\lim_{x \to c} g(x) \neq 0$, we know that:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

But this limit law does not tell us anything if $\lim_{x \to c} g(x) = 0$.

What if we had additional information? One might deduce that if $\lim_{x \to c} g(x) = 0$, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \pm\infty$$

However, this limit might not always exist, and is not necessarily infinite.

If we also knew that $\lim_{x\to c} f(x) = 0$, one might consider that $\lim_{x\to c} \frac{f(x)}{g(x)}$ may depend on how much faster f slams into 0 than g does. In fact, l'Hôpital's Rule gives us the following:

Theorem. *If both limits are either 0 or are both infinite, and* $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ *exists (which also presupposes that* f *and* g *are both differentiable on some neighborhood around* c*, and that* g' *must not be 0 in some neighborhood around* c*), then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

L'Hôpital's Rule then takes care of two ***indeterminate forms***:

$$\frac{0}{0} \quad \& \quad \pm\frac{\infty}{\infty}$$

(These are not equal to one! You know who you are!)

Example. Calcuate $\lim_{x\to\infty} \frac{2x+1}{e^x}$.

Proof. First, we observe that $\lim_{x\to\infty} 2x + 1 = \infty$ and that $\lim_{x\to\infty} e^x = \infty$. We can thus proceed to the next step of l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{2x+1}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

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Example. Calcuate $\lim_{x\to 1} \frac{x^a-1}{x^b-1}$.

First, we observe that $\lim_{x\to 1} x^a - 1 = 0$ and that $\lim_{x\to 1} x^b - 1 = 0$. Then:

$$\lim_{x\to 1} \frac{x^a-1}{x^b-1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$$

Sometimes it may be necessary to apply l'Hôpital's Rule more than once consecutively:

$$\lim_{x \to \infty} \frac{x^2}{e^{3x}} = \lim_{x \to \infty} \frac{2x}{3e^{3x}} = \lim_{x \to \infty} \frac{2}{9e^{3x}} = 0$$

Please take a minute to verify that the first two equalities are an appropriate application of l'Hôpital's Rule.

More indeterminate forms can benefit from the use of l'Hôpital's Rule:

$$0 \cdot (\pm\infty) \qquad 0^0 \qquad \infty^0 \qquad 1^\infty$$

The first can be taken care of by observing that $\frac{1}{\infty}$ can be treated as 0:

$$\lim fg = \lim \frac{f}{1/g} = \lim \frac{f'}{-g'/g^2} = -\lim \frac{f'g^2}{g'}$$

The indeterminant forms 0^0, ∞^0, and 1^∞ can be reduced to $0 \cdot (\pm\infty)$, using the fact that the exponential function is continuous and that $\lim_{x\searrow 0} \ln x = -\infty$, $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to\infty} \ln x = +\infty$:

$$\lim f^g = \lim e^{g \ln f} = e^{\lim(g \ln f)}$$

Here is an indeterminate $\infty - \infty$ limit that is a seemingly unlikely candidate for l'Hôpital's Rule:

$$\begin{aligned}
\lim_{x \nearrow \pi/2} (\sec x - \tan x) &= \lim_{x \nearrow \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \\
&= \lim_{x \nearrow \pi/2} \frac{1 - \sin x}{\cos x} \\
&= \lim_{x \nearrow \pi/2} \frac{-\cos x}{-\sin x} \\
&= 0
\end{aligned}$$