Theorem (Single Case of l'Hôpital's Rule)**.** *Let* f *and* g *be functions and suppose:*

- f *and* g *are both differentiable on some open interval* $(0, \lambda)$*, where* $\lambda > 0$*.*

- g *and* g' *are non-zero on* $(0, \lambda)$

- $\lim_{x \searrow 0} f(x) = \lim_{x \searrow 0} g(x) = 0$

- $\lim_{x \searrow 0} \frac{f'(x)}{g'(x)}$ *exists.*

Then $\lim_{x \searrow 0} \frac{f(x)}{g(x)} = \lim_{x \searrow 0} \frac{f'(x)}{g'(x)}$.

Proof. Let $L = \lim_{x \searrow 0} \frac{f'(x)}{g'(x)}$ and $\varepsilon > 0$. By the ε–δ definition of limit, we can freely choose a $\delta > 0$ so that given any x in the interval $(0, \delta)$, $\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon$. That is, we can make $\frac{f'(x)}{g'(x)}$ as close as we wish to the limit L as we desire.

Now consider some arbitrary x in the interval $(0,\delta)$. We observe that f and g are continuous $[0,x]$ and differentiable on $(0,x)$, so if we consider the function $h : [0,x] \to \mathbb{R}$ defined by:

$$h(t) = \begin{cases} 0 & \text{if } t = 0 \\ f(x)\,g(t) - g(x)\,f(t) & \text{otherwise} \end{cases}$$

Observe that since $\lim_{t \searrow 0} f(t) = \lim_{t \searrow 0} g(t) = 0$, previously established limit laws tell us that $\lim_{t \searrow 0} h(t) = 0$, so h is also continuous $[0,x]$ and differentiable on $(0,x)$, with h' given by:

$$h'(t) = f(x)\,g'(t) - g(x)\,f'(t)$$

*

Moreover, we observe that $h(x) = 0 = h(0)$.

*(Observe that $f(x)$ and $g(x)$ are actually constants with respect to t.)

By Rolle's Theorem, there exists a c in the interval $(0, x)$ such that $h'(c) = 0$.

But then this means that $f(x)\,g'(c) - g(x)\,f'(c) = 0$, and with some algebraic manipulation, we arrive at:

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

But since $0 < c < x < \delta$, we must have:

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon$$

This implies that $\lim_{x \searrow 0} \frac{f(x)}{g(x)} = \lim_{x \searrow 0} \frac{f'(x)}{g'(x)}$. ❑

Theorem ($\frac{\infty}{\infty}$ Case of l'Hôpital's Rule)**.** *Let $c \in \mathbb{R}$ and suppose the following about functions f and g:*

- *f and g are both differentiable on some open interval $(0, \lambda)$, where $\lambda > 0$.*

- *g and g' are non-zero on $(0, \lambda)$*

- $\lim_{x \searrow 0} f(x) = \lim_{x \searrow 0} g(x) = \pm\infty$

- $\lim_{x \searrow 0} \frac{f'(x)}{g'(x)}$ *exists and is finite.*

Then $\lim_{x \searrow 0} \frac{f(x)}{g(x)} = \lim_{x \searrow 0} \frac{f'(x)}{g'(x)}$.

Proof. Let $L = \lim_{x \searrow 0} \frac{f'(x)}{g'(x)}$ and $\varepsilon > 0$. By the ε–δ definition of limit, we can freely choose a $\delta > 0$ so that given any x in the interval $(0, \delta)$, $\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$. That is, we can make $\frac{f'(x)}{g'(x)}$ as close as we wish to the limit L as we desire.

Now consider some arbitrary x in the interval $(0, \delta)$. We observe that f and g are continuous $[0, x]$ and differentiable on $(0, x)$, so if we consider the function $h : [0, x] \to \mathbb{R}$ defined by:

$$h(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{f(x)}{g(t)} - \frac{g(x)}{f(t)} & \text{otherwise} \end{cases}$$

Observe that since $\lim_{t \searrow 0} f(t) = \lim_{t \searrow 0} g(t) = 0$, previously established limit laws tell us that $\lim_{t \searrow 0} h(t) = 0$, so h is also continuous $[0, x]$ and differentiable on $(0, x)$, with h' given by:

$$h'(t) = f(x) g'(t) - g(x) f'(t)$$

†

Moreover, we observe that $h(x) = 0 = h(0)$.

†(Observe that $f(x)$ and $g(x)$ are actually constants with respect to t.)

By Rolle's Theorem, there exists a c in the interval $(0, x)$ such that $h'(c) = 0$.

But then this means that $f(x)\,g'(c) - g(x)\,f'(c) = 0$, and with some algebraic manipulation, we arrive at:

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

But since $0 < c < x < \delta$, we must have:

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon$$

This implies that $\lim_{x \searrow 0} \frac{f(x)}{g(x)} = \lim_{x \searrow 0} \frac{f'(x)}{g'(x)}$. ❑