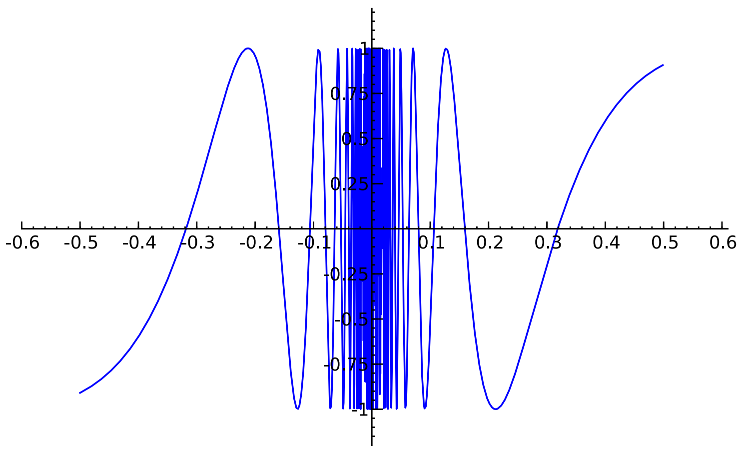
SPOTLIGHT PROBLEMS – WEEK 2

EXERCISE 2.1 (Topologist's Sine Curve). Let $f : \mathbb{R}^* \to [0, 1]$ be a function defined by:

$$f(x) = \sin \frac{1}{x}$$

The graph of which is:



(1) Find two sequences $(x_1, x_2, x_3, \dots)$ and $(y_1, y_2, y_3, \dots)$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, but $f(x_n) = 1$ and $f(y_n) = -1$ for all $n \in \mathbb{N}$. (Hint: Use the fact that for an integer n, $\sin\left(\frac{\pi}{2} + n\pi\right) = 1$ if n is even, and $\sin\left(\frac{\pi}{2} + n\pi\right) = -1$ if n is odd.)

(2) Deduce whether or not $\lim_{x\to 0} \sin \frac{1}{x}$ exists, and justify your conclusion.

EXERCISE 2.2 (Limits at infinity). Assume the limit laws on page 51 of your text also hold for calculating limits at infinity. Calculate the following limits, using algebraic manipulations as necessary:

(1) $\lim_{x\to\infty} \frac{x+3}{2-x}$
(2) $\lim_{x\to\infty} \frac{x^2+2x-1}{3+3x^2}$
(3) $\lim_{x\to\infty} \frac{x^2+4}{x+3}$
(4) $\lim_{x\to\infty} \frac{2x^3-16x^2}{4x^2+3x^3}$
(5) $\lim_{x\to\infty} \frac{x^4+3x}{x^4+2x^5}$
(6) $\lim_{x\to\infty} \frac{3e^x+2}{2e^x+3}$
(7) $\lim_{x\to\infty} \frac{2e^{-x}+3}{3e^{-x}+2}$

EXERCISE 2.3. If $p : \mathbb{R} \to \mathbb{R}$ is an n-th degree polynomial function given by:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

Show that:

$$\lim_{x\to+\infty} \frac{p(x)}{a_n x^n} = 1 \text{ and } \lim_{x\to-\infty} \frac{p(x)}{a_n x^n} = 1$$

Next, show that if n is odd, there exists a $c \in \mathbb{R}$ such that $p(c) = 0$.

(Hint: Use the fact $\lim_{x\to+\infty} \frac{p(x)}{a_n x^n} = 1$ implies you can find a positive real number x_+ such that $\frac{p(x_+)}{a_n x_+^n}$ is within $\varepsilon = \frac{1}{2}$ of 1, and that $\lim_{x\to-\infty} \frac{p(x)}{a_n x^n} = 1$ implies you can also find a negative real number x_- such

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that $\frac{p(x_-)}{a_n x_-^n}$ is within $\varepsilon = \frac{1}{2}$ of 1. You may also assume all polynomial functions are continuous, and that the Intermediate Value Theorem is true.)

Exercise 2.4 (Limit laws for constants, the identity function, and scalar multiplication)**.** Prove the following limits for arbitrary $c \in \mathbb{R}$:

- If k is a constant, then $\lim_{x\to c} k = k$.
- $\lim_{x\to c} x = c$
- If k is a constant, and f is a function such that $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$.
  - (Hint: The case that $k = 0$ can be reduced to the first limit. Then for $k \neq 0$ and an arbitrary $\varepsilon > 0$, you may freely choose $\delta > 0$ so that values of $f(x)$ are within $\frac{\varepsilon}{|k|}$ of $\lim_{x\to c} f(x)$ whenever x is within δ of c, but not equal to c.)

What do these limits imply about the continuity of constant functions, the identity function $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ defined by $\mathrm{id}_{\mathbb{R}}(x) = x$, and continuity of kf as it relates to the continuity of f?

Exercise 2.5 (Limit law for addition)**.** Prove the addition rule for limits, using the ε–δ definition for limit:

Let $f, g : D \to \mathbb{R}$ be functions, let $c \in \mathbb{R}$. Now suppose that the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. Then the limit $\lim_{x\to c} (f+g)(x)$ also exists, and moreover, is equal to $\lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.

(Hint: You want to choose a δ so that you can place $f(x)$ within $\frac{\varepsilon}{2}$ of $\lim_{x\to c} f(x)$, and $g(x)$ within $\frac{\varepsilon}{2}$ of $\lim_{x\to c} g(x)$.)

What does this imply about the continuity of $f + g$ as it relates to the continuity of f and g?

Exercise 2.6 (Limit law for multiplcation)**.** Prove a special case of the multiplication rule for limits, using the ε–δ definition for limit:

Let $f, g : D \to \mathbb{R}$ be functions, let $c \in \mathbb{R}$. Now suppose that the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, and are equal to 0. Then the limit $\lim_{x\to c} fg(x)$ also exists, and moreover, is also equal to 0.

(Hint: You want to choose a δ so that you can place $f(x)$ within $\varepsilon^* = \min\{\varepsilon, 1\}$ of $\lim_{x\to c} f(x)$, and $g(x)$ within ε^* of $\lim_{x\to c} g(x)$.)

Next, assume the limits $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ still exist, but are not necessarily zero. Verify the equality:

$$fg = \left(f - \lim_{x\to c} f(x)\right)\left(g - \lim_{x\to c} g(x)\right) + f \cdot \lim_{x\to c} g(x) + g \cdot \lim_{x\to c} f(x) - \lim_{x\to c} f(x) \lim_{x\to c} g(x)$$

(Hint: $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are constants here, and you may assume that the limit laws for constants, addition, and scalar multiplication, as listed above, are true.)

What does this imply about the continuity of fg as it relates to the continuity of f and g?

Exercise 2.7 (Limit law for division)**.** Prove a special case for the division rule for limits, using the ε–δ definition for limit:

Let $f : D \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$. Now suppose that the limit $\lim_{x\to c} f(x)$ exists and is non-zero. Then the limit $\lim_{x\to c} \frac{1}{f(x)}$ also exists, and moreover, is equal to $\frac{1}{\lim_{x\to c} f(x)}$.

(Hint: You want to choose a δ so that you can place $f(x)$ within $\varepsilon^* = \min\left\{\frac{\varepsilon\cdot(\lim_{x\to c} g(x))^2}{2}, \frac{|\lim_{x\to c} g(x)|}{2}\right\}$ of $\lim_{x\to c} f(x)$, and $g(x)$ within ε^* of $\lim_{x\to c} g(x)$.)

What does this imply about the continuity of $\frac{1}{f}$ as it relates to the continuity of f?

Exercise 2.8 (Advanced problem of the week)**.** Rigorously prove this particular version of the Intermediate Value Theorem:

Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous everywhere and suppose that $f(a) < 0 < f(b)$. Then there exists a c with $a \le c \le b$ such that $f(c) = 0$.

(Hint: This proof requires the least upper bound property and the ε–δ criterion of continuity.)