SPOTLIGHT PROBLEMS – WEEK 3

Exercise 3.1. The Heaviside step function, $H : \mathbb{R} \to \mathbb{R}$, is given by:

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Observe that H is continuous everywhere except at 0.

Now let $f, g, h : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) = t^2$, $g(t) = -t^2$ and $h(t) = 0$, for all t.

(1) Graph H.
(2) Write down formulas for $H \circ f$, $H \circ g$, and $H \circ h$ and graph each of these functions.
(3) Calculate the following:
  (a) $\lim_{t\to 0} H(f(t))$ and $H(\lim_{t\to 0} f(t))$;
  (b) $\lim_{t\to 0} H(g(t))$ and $H(\lim_{t\to 0} g(t))$;
  (c) $\lim_{t\to 0} H(h(t))$ and $H(\lim_{t\to 0} h(t))$;

Exercise 3.2 (Hours of Daylight as a Function of Latitude). Let $S : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^+$ be a function mapping a line of latitude in the northern hemisphere to the number of sunlight hours at that latitude on June 21.

(1) What does $S(0)$ represent? What is its actual value?
(2) Let x_0 be the latitude of the Arctic Circle, approximately $66°33'39''$, or 1.1617 radians. For some constants a and b, S is given by:

$$S(x) = \begin{cases} a + b \arcsin \frac{\tan x}{\tan x_0} & \text{if } 0 \le x < x_0 \\ 24 & \text{if } x_0 \le x \le \frac{\pi}{2} \end{cases}$$

Find a and b so that $S(0)$ agrees with your answer in 1, and S is continuous.
(3) Graph S.
(4) Does S appear to be differentiable on $\left(0, \frac{\pi}{2}\right)$?

Exercise 3.3. Let f be a function differentiable on an open interval (a, b), and suppose that f attains a maximum at some point c in the interval (a, b). That is, $f(x) \le f(c)$ for all x in the interval (a, b).

(1) Draw a possible graph of f.
(2) Show that $f'(c) = 0$. (Hint: Consider $\lim_{h\nearrow 0} \frac{f(c+h)-f(c)}{h}$ and $\lim_{h\searrow 0} \frac{f(c+h)-f(c)}{h}$.

Exercise 3.4 (Surface area of a sphere). Integral calculus can be used to show that the volume of a sphere as related to its radius is modeled by a function V defined by $V(r) = \frac{4}{3}\pi r^3$.

Consider two spheres with the same center, one with radius r, the other with radius $r + h$, with h some small amount. Let $S(r)$ represent an unknown formula for the surface area of the sphere with radius r, and consider the volume of the space between the two boundaries of the spheres in two different ways:

- As being the difference of the volumes of the spheres.
- As an approximation involving $S(r)$ and h.

Assuming that the approximation becomes better as h is smaller:

(1) Write $S(r)$ as a limit in terms of V, r, and h.
(2) Algebraically determine this limit.

Date: July 6, 2009.