SPOTLIGHT PROBLEMS – WEEK 4

Exercise 4.1 (Acceleration due to gravity). Newton's law of gravitation states that the gravitational force between two objects is jointly proportional to their masses and inversely proportional to the square of the distance between the objects:

$$F(m_1, m_2, r) = \frac{Gm_1m_2}{r^2}$$

Where G, informally known as "Big G," is known as the Gravitational constant.

Newton's second law of motion states that the relationship between a force $\vec{\mathbf{F}}$, mass m, and acceleration $\vec{\mathbf{a}}$, is given by:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

(1) Search online for a decimal approximation for G. Include units.
(2) Assume you are on a planet with mass m_1, and radius r. This time, consider the "grapefruit-in-flight" problem, where you are standing at the edge of a platform h_0 meters above the surface of the planet, holding a grapefruit with mass m_2 in your hand.
   (a) Determine the gravitational force between the grapefruit and the planet. Due to scale, you may assume the distance between the grapefruit and the center of the planet stays constant at r to achieve an acceptable approximation.
   (b) Determine the acceleration due to this gravitational force. (This quantity is known as the surface gravity.)
   (c) Suppose you launch the grapefruit almost vertically at an initial velocity of $\vec{\mathbf{v}}_0$. Find a function modelling the height of the grapefruit above the ground. Due to scale, you may assume acceleration due to gravity is constant. (Hint: Consider looking for a second-degree polynomial function $\vec{\mathbf{h}}$ whose second derivative is identically equal to your answer in (b), initial velocity is $\vec{\mathbf{v}}_0$, and whose initial height is h_0.)
(3) Research the masses and radii of the eight[1] planets. For each planet, assume your platform is 100 meters above the surface, and that the initial velocity of grapefruit upon launch is 20 meters per second.
   (a) Determine the height function $\vec{\mathbf{h}}$.
   (b) Determine when the grapefruit attains its maximum height.
   (c) Determine the maximum height attained.
   (d) Determine when the grapefruit hits the ground.[2]
   (e) Determine the velocity of the grapefruit when it hits the ground.

Be careful with your unit calculations (e.g., kilometers vs. meters). Please cite your sources of your researched values.

Exercise 4.2 (Derivatives of even and odd functions). Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove the following using the difference quotient definition of derivative:

(1) If f is an odd function, then f' is an even function.
(2) If f is an even function, then f' is an odd function.

Exercise 4.3 (Leibniz rule). The Binomial Theorem states that given any real numbers a and b, and a natural number n:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

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[1] Pluto is no longer considered a planet as of 2006, but if you wish to include Pluto, that is fine by me. You may even include some of the planets' satellites if you wish.

[2] Assume for the interest of this problem that the four gas giants are solid.

A surprisingly similar result is that provided n-times differentiable functions f and g, the n-th derivative of their product is given by:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

Prove this equality using mathematical induction.

(For the benefit of students in MTH 251 who may be unfamiliar with summation notation, binomial coefficients, and mathematical induction, please include explainations of these three ideas, which should come before the actual proof. You might also relate binomial coefficients to Pascal's Triangle, and mention Pascal's Rule.)

Exercise 4.4 (n-th derivative of power functions)**.** Show, by using mathematical induction, that $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n}$, whenever $n \leq m$, and $\frac{d^n}{dx^n}(x^m) = 0$, whenever $n > m$.

Exercise 4.5 (Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on the ***closed*** interval $[a, b]$ and differentiable on the ***open*** interval (a, b), and suppose that $f(a) = f(b)$. Rolle's Theorem states that there is a c in the open interval (a, b) such that $f'(c) = 0$. (Hint: Use the Extreme Value Theorem, and the result to Spotlight Problem 3.3.)

Next, generalize this result to the Mean Value Theorem, which relaxes the condition that $f(a) = f(b)$, and instead states that there is some c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. (Hint: Consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$.)

Exercise 4.6 (Relationship between positive derivative and increasing function)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on the ***closed*** interval $[a, b]$ and differentiable on the ***open*** interval (a, b). Show that if f' is positive on (a, b), then f is increasing on $[a, b]$. (Hint: Use Mean Value Theorem.)

Exercise 4.7 (Advanced problem of the week)**.** This problem proposes a more potent test for determining when a differentiable function attains a local extremum, a generalization of the second derivative test, which is inconclusive when we find that $f''(c) = f'(c) = 0$.

Let $f : D \to \mathbb{R}$ be m times differentiable on some open interval containing c, and suppose that $f^{(m)}$ is also continuous at c.

(1) Show that if $f^{(n)}(c) = 0$ for all $1 \leq n < m$, and $f^{(m)}(c) \neq 0$, then there is some function $h : \mathbb{R} \to \mathbb{R}$, continuous on the interval containing c, such that:

$$f'(x) = (x - c)^{m-1} h(x), \forall x \in \mathbb{R} \;\;\&\;\; h(c) \neq 0$$

(You may assume L'Hôpital's Rule to be true.)

(2) Assume that f satisfies the hypothesis in (2). Show the following:
  (a) If m is even and $f^{(m)}(c) > 0$, then f has a local minimum at c.
  (b) If m is even and $f^{(m)}(c) < 0$, then f has a local maximum at c.
  (c) If m is odd and $f^{(m)}(c) > 0$, then f is increasing on some interval containing c.
  (d) If m is odd and $f^{(m)}(c) < 0$, then f is decreasing on some interval containing c.

To make this problem somewhat simpler, you may, without loss of generality, let $c = 0$.