SPOTLIGHT PROBLEMS – WEEK 5

Exercise 5.1. The power rule along with the chain rule states that given a differentiable function f and integer n, $(f^n)' = nf^{n-1}f'$. Combining the derivative of the exponential function with the chain rule gives us $\left(e^f\right)' = f'e^f$.

Letting f and g be differentiable functions, with $f > 0$, find a formula for $(f^g)'$.

Exercise 5.2 (Leibniz rule)**.** The Binomial Theorem states that given any real numbers a and b, and a natural number n:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

A surprisingly similar result is that provided n-times differentiable functions f and g, the n-th derivative of their product is given by:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

Prove this equality using mathematical induction.

Exercise 5.3 (n-th derivative of power functions)**.** Show, by using mathematical induction, that $\frac{d^n}{dx^n}\left(x^m\right) = \frac{m!}{(m-n)!}x^{m-n}$, whenever $n \leq m$, and $\frac{d^n}{dx^n}\left(x^m\right) = 0$, whenever $n > m$.

Exercise 5.4 (Relationship between positive derivative and increasing function)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on the ***closed*** interval $[a, b]$ and differentiable on the ***open*** interval (a, b). Show the following:

- If f' is positive on (a, b), then f is increasing on $[a, b]$.
- If $f' = 0$ on (a, b), then f is constant on $[a, b]$.

(Hint: Use Mean Value Theorem.)

Exercise 5.5 (Advanced problem of the week)**.** This problem proposes a more potent test for determining when a differentiable function attains a local extremum, a generalization of the second derivative test, which is inconclusive when we find that $f''(c) = f'(c) = 0$.

Let $f : D \to \mathbb{R}$ be m times differentiable on some open interval containing c, and suppose that $f^{(m)}$ is also continuous at c.

(1) Show that if $f^{(n)}(c) = 0$ for all $1 \leq n < m$, and $f^{(m)}(c) \neq 0$, then there is some function $h : \mathbb{R} \to \mathbb{R}$, continuous on the interval containing c, such that:

$$f'(x) = (x-c)^{m-1} h(x), \forall x \in \mathbb{R} \;\; \& \;\; h(c) \neq 0$$

(You may assume L'Hôpital's Rule to be true.)

(2) Assume that f satisfies the hypothesis in (2). Show the following:
   (a) If m is even and $f^{(m)}(c) > 0$, then f has a local minimum at c.
   (b) If m is even and $f^{(m)}(c) < 0$, then f has a local maximum at c.
   (c) If m is odd and $f^{(m)}(c) > 0$, then f is increasing on some interval containing c.
   (d) If m is odd and $f^{(m)}(c) < 0$, then f is decreasing on some interval containing c.

To make this problem somewhat simpler, you may, without loss of generality, let $c = 0$.

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