SPOTLIGHT PROBLEMS – WEEK 6

Exercise 6.1 (Taylor Polynomials)**.** Given a differentiable function f, and c in the domain of f, the linear approximation of f around c is given by:

$$p_1(x) = f'(c)(x-c) + f(c)$$

Or, more verbosely,

$$p_1(x) = f'(c)(x-c)^1 + f(c)(x-c)^0$$

Now suppose we are interested in finding a best-fit quadratic polynomial, p_2, for a twice-differentiable function f, centered at c:

$$p_2(x) = a_2(x-c)^2 + a_1(x-c) + a_0$$

Find a_0, a_1, and a_2 in terms of f and its derivatives so that $p_2(c) = f(c)$, $p_2'(c) = f'(c)$, and $p_2''(c) = f''(c)$.

Now do the same for polynomials p_3 and p_4 of degrees 3 and 4 of similar form, and observe a pattern.

These polynomials are defined to be the Taylor polynomials of f centered at c. What do you expect the formula for the n-th degree Taylor polynomial to be?

Meanwhile, the Taylor *series* is given by the infinite sum $\sum_{k=0}^{\infty} a_k (x-c)^k$, where the coefficients a_k are derived in the same way as for Taylor polynomials. Find the Taylor series for the natural exponential function, sine, and cosine, taking $c = 0$.

Exercise 6.2 (Racetrack Principle)**.** Using the Increasing Function Theorem (Spotlight Problem #5.4), prove the Racetrack Principle:

Let f and g are continuous on an interval $[a, b]$ differentiable on (a, b), with $f' \leq g'$ on (a, b):

- If $f(a) \leq g(a)$, then $f \leq g$ on $[a, b]$.
- If $g(b) \leq f(b)$, then $g \leq f$ on $[a, b]$.

Exercise 6.3 (Optimization)**.** Find all local and global extremum of the given functions:

(1) $f : [-1, 1] \to \mathbb{R}$, defined by $f(x) = x^3$
(2) $g : [-1, 1] \to \mathbb{R}$, defind by $g(x) = \sqrt[3]{x}$
(3) $h : [-1, 1] \to \mathbb{R}$, defind by $h(x) = x^3(x-1)$
(4) $j : [-1, 1] \to \mathbb{R}$, defined by $j(x) = 2|x|(|x| - 2) + 1$
(5) $k : (-1, 1] \to \mathbb{R}$, defined by $k(x) = x^5 + x$

Try finding the extremum ***before*** graphing the functions.

Exercise 6.4 (Power rule for negative integers and rational exponents)**.** Until now, we have assumed that the power rule holds true for all real exponents, but the sketch of the proof provided in class only proves the case of a natural number exponent ($\mathbb{N} = \{0, 1, 2, 3, \dots\}$).

- Using the quotient rule and the power rule for exponents in $\mathbb{N}$, show the power rule also holds for all of $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. (Start by letting m be a negative integer, and write $x^m = \frac{1}{x^{-m}}$.)
- Next, prove the power rule for exponents in $\mathbb{Q}$, using the chain rule, and assuming that the power rule works for all exponents in $\mathbb{Z}$. (Start with the equation $\left(x^{p/q}\right)^q = x^p$.)

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