SPOTLIGHT PROBLEMS – WEEK 7

Exercise 7.1 (Taylor Polynomials). Given a differentiable function f, and c in the domain of f, the linear approximation of f around c is given by:

$$p_1(x) = f'(c)(x-c) + f(c)$$

Or, more verbosely,

$$p_1(x) = f'(c)(x-c)^1 + f(c)(x-c)^0$$

Now suppose we are interested in finding a best-fit quadratic polynomial, p_2, for a twice-differentiable function f, centered at c:

$$p_2(x) = a_2(x-c)^2 + a_1(x-c) + a_0$$

Find a_0, a_1, and a_2 in terms of f and its derivatives so that $p_2(c) = f(c)$, $p_2'(c) = f'(c)$, and $p_2''(c) = f''(c)$.

Now do the same for polynomials p_3 and p_4 of degrees 3 and 4 of similar form, and observe a pattern.

These polynomials are defined to be the Taylor polynomials of f centered at c. What do you expect the formula for the n-th degree Taylor polynomial to be?

Meanwhile, the Taylor *series* is given by the infinite sum $\sum_{k=0}^{\infty} a_k (x-c)^k$, where the coefficients a_k are derived in the same way as for Taylor polynomials. Find the Taylor series for the natural exponential function, sine, and cosine, taking $c = 0$.

Exercise 7.2 (Optimization). Find all local and global extremum of the given functions:

(1) $f : [-1,1] \to \mathbb{R}$, defined by $f(x) = x^3$
(2) $g : [-1,1] \to \mathbb{R}$, defind by $g(x) = \sqrt[3]{x}$
(3) $h : [-1,1] \to \mathbb{R}$, defind by $h(x) = x^3(x-1)$
(4) $j : [-1,1] \to \mathbb{R}$, defined by $j(x) = 2|x|(|x|-2)+1$
(5) $k : (-1,1] \to \mathbb{R}$, defined by $k(x) = x^5 + x$

Try finding the extremum ***before*** graphing the functions.

Exercise 7.3 (Power rule for negative integers and rational exponents). Until now, we have assumed that the power rule holds true for all real exponents, but the sketch of the proof provided in class only proves the case of a natural number exponent ($\mathbb{N} = \{0,1,2,3,\dots\}$).

- Using the quotient rule and the power rule for exponents in $\mathbb{N}$, show the power rule also holds for all of $\mathbb{Z} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$. (Start by letting m be a negative integer, and write $x^m = \frac{1}{x^{-m}}$.)
- Next, prove the power rule for exponents in $\mathbb{Q}$, using the chain rule, and assuming that the power rule works for all exponents in $\mathbb{Z}$. (Start with the equation $\left(x^{p/q}\right)^q = x^p$.)

Exercise 7.4 (Leibniz rule). The Binomial Theorem states that given any real numbers a and b, and a natural number n:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

A surprisingly similar result is that provided n-times differentiable functions f and g, the n-th derivative of their product is given by:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

Prove this equality using mathematical induction.

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Exercise 7.5 (n-th derivative of power functions)**.** Show, by using mathematical induction, that $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n}$, whenever $n \leq m$, and $\frac{d^n}{dx^n}(x^m) = 0$, whenever $n > m$.

Exercise 7.6 (Linear least squares regression)**.** Suppose you have the following data from a sample:

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$$

Linear least squares regression is the modelling of a sample of data with a linear function (not to be confused with linear approximation!).

(1) Define ℓ by $\ell(x) = mx + b$, and suppose m and b are chosen so that $\sum_{k=1}^{n} (\ell(x_k) - y_k)^2$ is minimized (hence, least squares). Reduce the problem of finding m and b to minimize the sum to a system of two equations.
(2) Find the linear least squares regression of the following data:

x_k	y_k
2	8.7
5	140.3
7	355.8
8	539.2

Exercise 7.7 (Advanced problem of the week)**.** This problem proposes a more potent test for determining when a differentiable function attains a local extremum, a generalization of the second derivative test, which is inconclusive when we find that $f''(c) = f'(c) = 0$.

Let $f : D \to \mathbb{R}$ be m times differentiable on some open interval containing c, and suppose that $f^{(m)}$ is also continuous at c.

(1) Show that if $f^{(n)}(c) = 0$ for all $1 \leq n < m$, and $f^{(m)}(c) \neq 0$, then there is some function $h : \mathbb{R} \to \mathbb{R}$, continuous on the interval containing c, such that:

$$f'(x) = (x-c)^{m-1} h(x), \forall x \in \mathbb{R} \;\; \& \;\; h(c) \neq 0$$

(You may assume L'Hôpital's Rule to be true.)
(2) Assume that f satisfies the hypothesis in (2). Show the following:
  (a) If m is even and $f^{(m)}(c) > 0$, then f has a local minimum at c.
  (b) If m is even and $f^{(m)}(c) < 0$, then f has a local maximum at c.
  (c) If m is odd and $f^{(m)}(c) > 0$, then f is increasing on some interval containing c.
  (d) If m is odd and $f^{(m)}(c) < 0$, then f is decreasing on some interval containing c.

To make this problem somewhat simpler, you may, without loss of generality, let $c = 0$.

Exercise 7.8. Prove the following propositions:

(1) (Bounded Monotone Convergence Theorem) Let $(x_k)_{k\in\mathbb{N}}$ be an increasing sequence ($j > k$ implies $x_j \geq x_k$ for all j and k) that is bounded from above by some M ($x_k \leq M$ for all k). Then $\lim_{k\to\infty} x_k$ exists and is finite.
(2) (Monotonicity of sup and inf) Let $\emptyset \neq B \subseteq A \subseteq \mathbb{R}$. If A has an upper bound, then $\sup B \leq \sup A$. If A has a lower bound, then $\inf A \leq \inf B$.
(3) (Heine Definition of Continuity) Let f be continuous at some $c \in \mathbb{R}$ and let $(x_k)_{k\in\mathbb{N}}$ be a sequence such that $\lim_{k\to\infty} x_k = c$. Then $\lim_{k\to\infty} f(x_k) = f(c)$.
(4) (Extreme Value Theorem) Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists a c and d in $[a,b]$ such that for all x in $[a,b]$, $f(d) \leq f(x) \leq f(c)$.

    You may assume the existence of a global minimum follows as a corollary from the existence from a global maximum. Furthermore, there are two items to prove here:
  (a) That f is bounded from above. (There is some $M \in \mathbb{R}$ such that $f(x) \leq M$ for all x in $[a,b]$.)
  (b) That f attains a maximum. (There is some c such that $f(c) = \sup\{f(x) \mid x \in [a,b]\}$.)